Algorithms for Approximate String Matching

Part I

Levenshtein Distance
Hamming Distance
Approximate String Matching with k Differences
Longest Common Subsequences

Part II

"A Fast and Practical Bit-Vector Algorithm for the Longest Common Subsequences Problem"

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Levenshtein Distance

Levenshtein distance is named after the Russian scientist Vladimir Levenshtein, who devised the algorithm in 1965. If you cannot spell or pronounce Levenshtein, the metric is also called *edit distance*.

The edit distance $\delta(p,t)$ between two strings p (pattern) and t (text) $(m = |p|, n = |t|)$ is the minimum number of insertions, deletions and replacements to make p equal to t.

- **[Insertion]** insert a new letter a into x. An insertion operation on the string $x = vw$ consists in adding a letter a, converting x into $x' = vaw$.
- **[Deletion]** delete a letter a from x. A deletion operation on the string $x = vaw$ consists in removing a letter, converting x into $x' = vw$.
- **[Replacement]** replace a letter a in x. A replacement operation on the string $x = vaw$ consists in replacing a letter for another, converting x into $x' = vbw$.

▶ Example 1:

$p =$"approximate_matching"
$t =$"appropiate_meaning"

```
           1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
String t:  a p p r o p r i a t  e  _  m  ε  e  a  n  i  n  g
           | | | | |       | |  |  |  |              |  |  |
String p:  a p p r o x i m a t  e  _  m  a  t  c  h  i  n  g
```

$\delta(t, p) = 7$

▶ Example 2:

$p =$"surgery"
$t =$"survey"

```
           1 2 3 4 5 6 7
String t:  s u r v e ε y
           | | |   |   |
String p:  s u r g e r y
```

$\delta(t, p) = 2$

▶ Solution: Using Dynamic Programing (DP): We need to compute a matrix $D[0..m, 0..n]$, where $D_{i,j}$ represents the minimum number of operations needed to match $p_{1..i}$ to $t_{1..j}$.

This is computed as follows:

$D[i, 0] = i$
$D[0, j] = j$
$D[i, j] = \min\{D[i-1, j] + 1, D[i, j-1] + 1, D[i, j] + \delta(p_i, t_j)\}$

$\delta(p, t) = D[m, n]$

▶ Pseudo-code:

```
procedure ED(p,t)     {m = |p|, n = |t|}
begin
    for i ← 0 to m do D[i,0] ← i
    for j ← 0 to n do D[0,j] ← j
    for i ← 1 to m do
        for j ← 1 to n do
            if p_i = t_j then D[i,j] ← D[i−1,j−1]
            else
                D[i,j] ← min(D[i,j−1], D[i−1,j], D[i−1,j−1]) + 1
        od
    od
    return D[m,n]
end
```

▶ Running time: $O(nm)$

▶ **Example:** p="survey", t="surgery"

		0	1	2	3	4	5	6	7
		ϵ	s	u	r	g	e	r	y
0	ϵ	0	1	2	3	4	5	6	7
1	s	1	0	1	2	3	4	5	6
2	u	2	1	0	1	2	3	4	5
3	r	3	2	1	0	1	2	3	4
4	v	4	3	2	1	1	2	3	4
5	e	5	4	3	2	2	1	2	3
6	y	6	5	4	3	3	2	2	**2**

	1	2	3	4	5	6	7
String t:	s	u	r	g	e	r	y
	|	|	|		|		|
String p:	s	u	r	v	e	ϵ	y

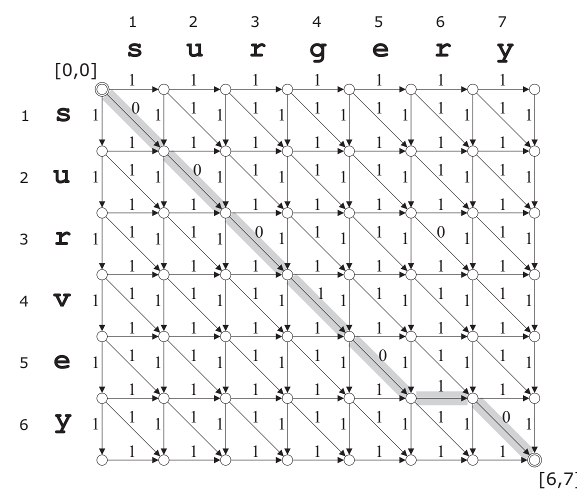
A Graph Reformulation
for the Edit Distance problem

The Dynamic Programming matrix D can be seen as a graph where the nodes are the cells and the edges represent the operations. The cost (weight) of the edges corresponds to the cost of the operations.

$\delta(t, p)$ = shortest-path from node [0,0] to the node [n,m].

▶ **Running time:** $O(nm \log(nm))$

▶ **Example:** p="survey", t="surgery"



Hamming Distance

The Hamming distance H is defined only for strings of the *same* length. For two strings p and t, $H(p,t)$ is the number of places in which the two strings differ, i.e., have different characters.

▶ **Examples:**

$H($"pinzon", "pinion"$) = 1$
$H($"josh", "jose"$) = 1$
$H($"here", "hear"$) = 2$
$H($"kelly", "belly"$) = 1$
$H($"AAT", "TAA"$) = 2$
$H($"AGCAA", "ACATA"$) = 3$
$H($"AGCACACA", "ACACACTA"$) = 6$

▶ **Pseudo-code:** too easy!!

▶ **Running time:** $O(n)$

Approximate String Matching with k Differences

▶ **Problem:** The k-*differences approximate string matching problem* is to find all occurrences of the *pattern* string p in the *text* string t with at most k differences (substitution, insertions, deletions).

▶ **Solution:** Using DP

$D[i,0] = i$
$D[0,j] = 0$
$D[i,j] = \min\{D[i-1,j]+1, D[i,j-1]+1, D[i,j]+\delta(p_i,t_j)\}$

if $D[m,j] \leq k$ then we say that p occurs at position j of t.

▶ Pseudo-code:

```
procedure KDifferences(p, t, k)      {m = |p|, n = |t|}
begin
    for i ← 0 to m do D[i, 0] ← i
    for j ← 0 to n do D[0, j] ← 0
    for i ← 1 to m do
        for j ← 1 to n do
            if p_i = t_j then D[i, j] ← D[i − 1, j − 1]
            else
                D[i, j] ← min(D[i, j − 1], D[i − 1, j], D[i − 1, j − 1]) + 1
        od
    od
    for j ← 0 to n do
        if D[m][j] ≤ k then Output(j)
    od
end
```

▶ Running time: $O(nm)$

▶ Example:

$p =$"CDDA",
$t =$"CADDACDACDBACBA"
$k = 1$

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		ϵ	C	A	D	D	A	C	D	A	C	D	B	A	C	B	A
0	ϵ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	C	1	0	1	1	1	1	0	1	1	0	1	1	1	0	1	1
2	D	2	1	1	1	1	2	1	0	1	1	0	1	2	1	1	2
3	D	3	2	2	1	1	2	2	1	1	2	1	1	2	2	2	2
4	A	4	3	2	2	2	**1**	2	2	**1**	2	2	2	**1**	2	3	2
							⇑			⇑				⇑			

p occurs in t ending at positions 5, 8 and 12.

Longest Common Subsequence

Preliminaries

For two sequences $x = x_1 \cdots x_m$ and $y_1 \cdots y_n$ $(n \geq m)$

we say that x is a *subsequence* of y and equivalently, y is a *supersequence* of x, if for some $i_1 < \cdots < i_p$, $x_j = y_{i_j}$.

Given a finite set of sequences, S, a *longest common subsequence* (LCS) of S is a longest possible sequence s such that each sequence in S is a supersequence of s.

Example: y=`"longest"`, x=`"large"`

```
String y:  l o n g e s t
           |     | |
String x:  l a r g e
```

LCS(y,x)=`"lge"`

Longest Common Subsequence

▶ **Problem:** The *Longest Common Subsequence* (LCS) of two strings, p and t, is a subsequence of both p and of t of maximum possible length.

▶ **Solution:** Using Dynamic Programing: We need to compute a matrix $L[0..m, 0..n]$, where $L_{i,j}$ represent the LCS for $p_{1..i}$ and $t_{1..j}$.

This is computed as follows:

$$L[i,j] = \begin{cases} 0, & \text{if either } i = 0 \text{ or } j = 0 \\ L[i-1, j-1] + 1, & \text{if } p_i = t_j \\ \max\{L[i-1, j], L[i, j-1]\}, & \text{if } p_i \neq t_j \end{cases}$$

▶ **Pseudo-code:**

```
procedure LCS(p, t)      {m = |p|, n = |t|}
begin
    for i ← 0 to m do L[i,0] ← 0
    for j ← 0 to n do L[0,j] ← 0
    for i ← 1 to m do
        for j ← 1 to n do
            if p_i = t_j then L[i,j] ← L[i−1, j−1] + 1
            else
                if L[i, j−1] > L[i−1, j] then L[i,j] ← L[i, j−1]
                else L[i,j] ← L[i−1, j]
        od
    od
    return L[m][n]
end
```

▶ **Running time:** $O(nm)$

▶ **Example 1:** $p =$"survey" and $t =$"surgery".

		0	1	2	3	4	5	6	7
		ϵ	s	u	r	g	e	r	y
0	ϵ	0	0	0	0	0	0	0	0
1	s	0	1	1	1	1	1	1	1
2	u	0	1	2	2	2	2	2	2
3	r	0	1	2	3	3	3	3	3
4	v	0	1	2	3	3	3	3	3
5	e	0	1	2	3	3	4	4	4
6	y	0	1	2	3	3	4	4	**5**

```
             1 2 3 4 5 6 7
String t:    s u r g e r y
             | | |   |  /
String p:    s u r v e y
```

LCS$(p, t) =$"surey"

LLCS$(p, t) = L[6, 7] = 5$

▶ **Example 2:**

$p =$"ttgatacatt"
$t =$"gaataagacc"

		0	1	2	3	4	5	6	7	8	9	10
		ϵ	g	a	a	t	a	a	g	a	c	c
0	ϵ	0	0	0	0	0	0	0	0	0	0	0
1	t	0	0	0	0	**1**	1	1	1	1	1	1
2	t	0	0	0	0	1	1	1	1	1	1	1
3	g	0	**1**	1	1	1	1	1	**2**	2	2	2
4	a	0	1	**2**	2	2	2	2	2	**3**	3	3
5	t	0	1	2	2	**3**	3	3	3	3	3	3
6	a	0	1	2	**3**	3	**4**	4	4	4	4	4
7	c	0	1	2	3	3	4	4	4	4	**5**	5
8	a	0	1	2	3	3	4	**5**	5	5	5	5
9	t	0	1	2	3	**4**	4	5	5	5	5	**5**

LCS$(p, t) =$?

LLCS$(p, t) = L[9, 10] = 5$

Part II

A Fast and Practical Bit-Vector Algorithm for the Longest Common Subsequence Problem

Some More Definitions

The ordered pair of *positions* i and j of L, denoted $[i, j]$, is a *match* iff $x_i = y_j$.

If $[i, j]$ is a match, and an LCS $s_{i,j}$ of $x_1x_2...x_i$ and $y_1y_2...y_j$ has length k, then k is the *rank* of $[i, j]$.

The match $[i, j]$ is k-*dominant* if it has rank k and for any other pair $[i', j']$ of rank k, either $i' > i$ and $j' \leq j$ or $i' \leq i$ and $j' > j$.

Computing the k-dominant matches is all that is needed to solve the LCS problem, since the LCS of x and y has length p iff the maximum rank attained by a dominant match is p.

A match $[i, j]$ *precedes* a match $[i', j']$ if $i < i'$ and $j < j'$.

Let r be the total number of matches points, and d be the total number of dominant points (all ranks). Then $0 \leq p \leq d \leq r \leq nm$.

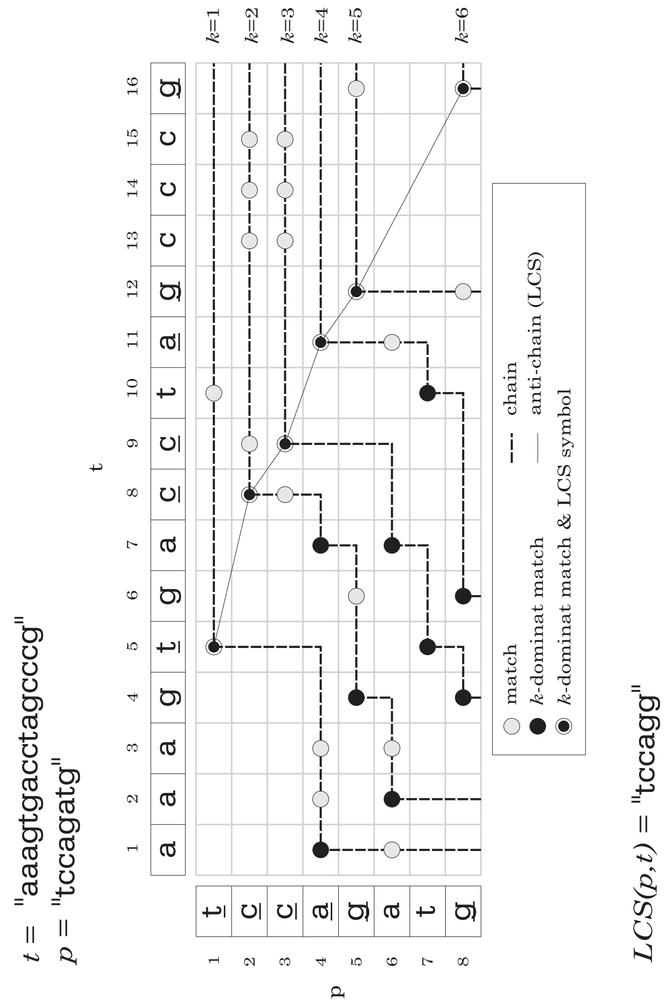
Let $\mathcal{R}$ denote a partial order relation on the set of matches in L.

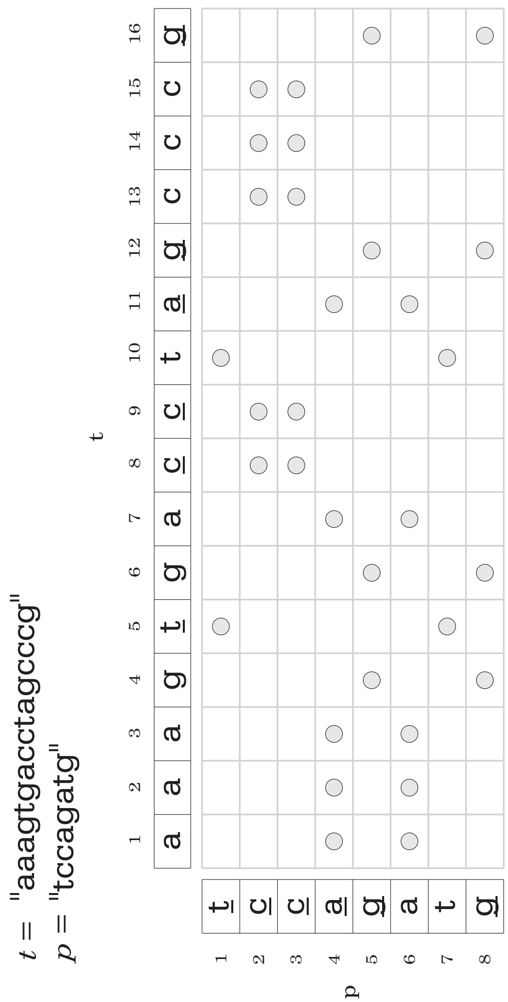
A set of matches such that in any pair one of the matches always precedes the other in $\mathcal{R}$ constitutes a *chain* relative to the partial order relation $\mathcal{R}$.

A set of matches such that in any pair neither element of the pair precedes the other in $\mathcal{R}$ constitutes an *antichain*.

Sankoff and Sellers (1973) observed that the LCS problem translates to finding a longest *chain* in the *poset* of matches induced by $\mathcal{R}$.

A decomposition of a poset into antichains partitions the poset into the minimum possible number of antichains.





A Simple Bit-Vector Algorithm

Here we will make use of word-level parallelism in order to compute the matrix L more efficiently.

The algorithm is based on the $O(1)$-time computation of each column in L by using a bit-parallel formula under the assumption that $m \leq w$, where w is the number of bits in a machine word or $O(nm/w)$-time for the general case.

An interesting property of the LCS allows to represent each column in L by using $O(1)$-space. That is, the values in the columns (rows) of L increase by at most one. i.e. $\Delta L[i,j] = L[i,j] - L[i-1,j] \in \{0,1\}$ for any $(i,j) \in \{1..m\} \times \{1..n\}$.

In other words ΔL will use the relative encoding of the dynamic programming table L.

$\Delta L'$ is defined as `NOT` ΔL.

Example: $x=$"ttgatacatt" and $y=$"gaataagacc".

		0	1	2	3	4	5	6	7	8	9	10
		ϵ	G	A	A	T	A	A	G	A	C	C
0	ϵ	0	0	0	0	0	0	0	0	0	0	0
1	T	0	0	0	0	**1**	1	1	1	1	1	1
2	T	0	0	0	0	1	1	1	1	1	1	1
3	G	0	**1**	1	1	1	1	1	**2**	2	2	2
4	A	0	1	**2**	2	2	2	2	2	**3**	3	3
5	T	0	1	2	2	**3**	3	3	3	3	3	3
6	A	0	1	2	**3**	3	**4**	4	4	4	4	4
7	C	0	1	2	3	3	4	4	4	4	**5**	5
8	A	0	1	2	3	3	4	**5**	5	5	5	5
9	T	0	1	2	3	**4**	4	5	5	5	5	**5**

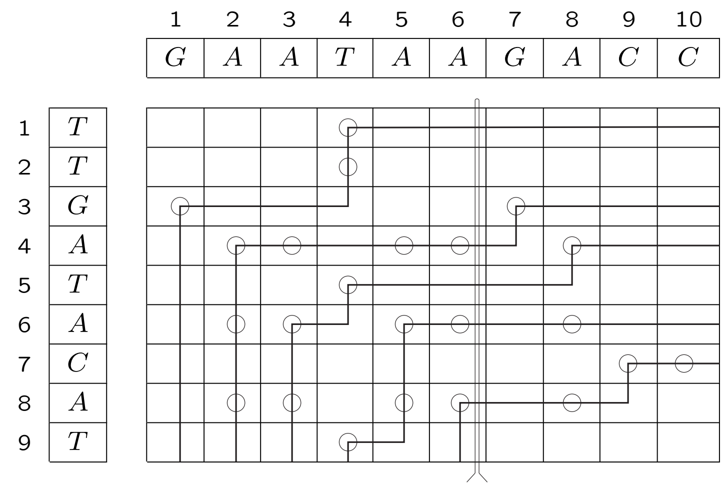
(a) Matrix L

		0	1	2	3	4	5	6	7	8	9	10
		ϵ	G	A	A	T	A	A	G	A	C	C
0	ϵ	0	0	0	0	0	0	0	0	0	0	0
1	T	0	0	0	0	1	1	1	1	1	1	1
2	T	0	0	0	0	0	0	0	0	0	0	0
3	G	0	1	1	1	0	0	0	1	1	1	1
4	A	0	0	1	1	1	1	1	0	1	1	1
5	T	0	0	0	0	1	1	1	1	0	0	0
6	A	0	0	0	1	0	1	1	1	1	1	1
7	C	0	0	0	0	0	0	0	0	0	1	1
8	A	0	0	0	0	0	0	1	1	1	0	0
9	T	0	0	0	0	1	0	0	0	0	0	0

(b) Matrix ΔL

		1	2	3	4	5	6	7	8	9	10
		G	A	A	T	A	A	G	A	C	C
1	T	1	1	1	0	0	0	0	0	0	0
2	T	1	1	1	1	1	1	1	1	1	1
3	G	0	0	0	1	1	1	0	0	0	0
4	A	1	0	0	0	0	0	1	0	0	0
5	T	1	1	1	0	0	0	0	1	1	1
6	A	1	1	0	1	0	0	0	0	0	0
7	C	1	1	1	1	1	1	1	1	0	0
8	A	1	1	1	1	1	0	0	0	1	1
9	T	1	1	1	0	1	1	1	1	1	1

Matrix $\Delta L'$



$\Delta L'_6 =< 011000101 >$

First we compute the array M of the vectors that result for each possible text character. If both the strings x and y range over the alphabet Σ then $M[\Sigma]$ is defined as $M[\alpha]_i = 1$ if $y_i = \alpha$ else 0.

Example: x= "ttgatacatt" and y= "gaataagacc".

		A	C	G	T
1	T	0	0	0	1
2	T	0	0	0	1
3	G	0	0	1	0
4	A	1	0	0	0
5	T	0	0	0	1
6	A	1	0	0	0
7	C	0	1	0	0
8	A	1	0	0	0
9	T	0	0	0	1

(a) Matrix M

		A	C	G	T
1	T	1	1	1	0
2	T	1	1	1	0
3	G	1	1	0	1
4	A	0	1	1	1
5	T	1	1	1	0
6	A	0	1	1	1
7	C	1	0	1	1
8	A	0	1	1	1
9	T	1	1	1	0

(a) Matrix M'

Basic steps of the algorithm

1. Computation of M and M'

2. Computation of matrix $\Delta L'$ (L) as follows:

$$L = \begin{cases} 2^m - 1, & \text{for } j = 0 \\ (L_{j-1} + (L_{j-1} \texttt{ AND } M(y_j))) \texttt{ OR } (L_{j-1} \texttt{ AND } M'(y_j)), & \text{for } j \in \{1..n\} \end{cases}$$

3. Let LLCS be the number of times a carry took place.

Pseudo-code

```
LLCS(x, y)    ▷ n = |y|, m = |x|, p = 0
begin
    ▷ Preprocessing
    for i ← 1 until m do
        M[α](i) ← y_i = α
        M'[α](i) ← y_i ≠ α
    ▷ Initialization
    L_0 = 2^m − 1
    ▷ TheMainStep
    for j ← 1 until n do
        L_j ← (L_{j−1} + (L_{j−1} AND M[y_j])) OR (L_{j−1} AND M'[y_j])
        if L_j(m + 1) = 1 then p++
    return p
end
```


Illustration of $\Delta L'_4$ Computation

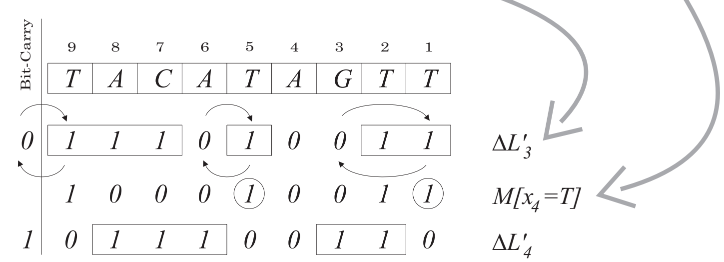
for x= "gaataagacc" and y= "ttgatacatt".

		0	1	2	3	4	5	6	7	8	9	10
		ϵ	G	A	A	T	A	A	G	A	C	C
1	T	1	1	1	1	0	0	0	0	0	0	0
2	T	1	1	1	1	1	1	1	1	1	1	1
3	G	1	0	0	0	1	1	1	0	0	0	0
4	A	1	1	0	0	0	0	0	1	0	0	0
5	T	1	1	1	1	0	0	0	0	1	1	1
6	A	1	1	1	0	1	0	0	0	0	0	0
7	C	1	1	1	1	1	1	1	1	1	0	0
8	A	1	1	1	1	1	1	0	0	0	1	1
9	T	1	1	1	1	0	1	1	1	1	1	1

(a) Matrix $\Delta L'$

		1	2	3	4
		A	C	G	T
1	T	0	0	0	1
2	T	0	0	0	1
3	G	0	0	1	0
4	A	1	0	0	0
5	T	0	0	0	1
6	A	1	0	0	0
7	C	0	1	0	0
8	A	1	0	0	0
9	T	0	0	0	1

(b) Matrix M



$$L_4 \leftarrow \overbrace{(\ \overbrace{L_3 + \overbrace{(L_3 \& M_T)}^{(1)}}^{(3)}\ ) \mid \overbrace{(L_3 \& M_T')}^{(2)}}^{(4)}$$

L_3		1	1	1	0	1	0	0	1	1	&
M_T		1	0	0	0	1	0	0	1	1	
(1)											

L_3		1	1	1	0	1	0	0	1	1	&
M_T'		0	1	1	1	0	1	1	0	0	
(2)											

L_3		1	1	1	0	1	0	0	1	1	+
(1)		1	0	0	0	1	0	0	1	1	
(3)											

(3)	1	0	1	1	1	0	0	1	1	0	|
(2)		0	1	1	0	0	0	0	0	0	
(4)											

$$L_4 \leftarrow \overbrace{(\ \overbrace{L_3 + \overbrace{(L_3 \& M_T)}^{(1)}}^{(3)}\ ) \mid \overbrace{(L_3 \& M_T')}^{(2)}}^{(4)}$$

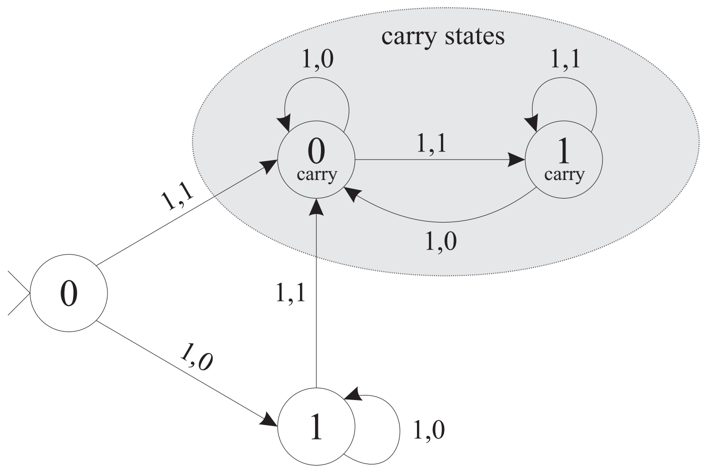
L_3		1	1	1	0	1	0	0	1	1	&
M_T		1	0	0	0	1	0	0	1	1	
(1)		1	0	0	0	1	0	0	1	1	

L_3		1	1	1	0	1	0	0	1	1	&
M_T'		0	1	1	1	0	1	1	0	0	
(2)		0	1	1	0	0	0	0	0	0	

L_3		1	1	1	0	1	0	0	1	1	+
(1)		1	0	0	0	1	0	0	1	1	
(3)	1	0	1	1	1	0	0	1	1	0	

(3)	1	0	1	1	1	0	0	1	1	0	|
(2)		0	1	1	0	0	0	0	0	0	
(4)	1	0	1	1	1	0	0	1	0	0	

Automata for Addition



Experimental Results