Linear Classification Models

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Content

Outline

Classification problems

- $predict(x) = \begin{cases} C_1, & y(x) \geq \text{threshold} \\ C_2, & y(x) < \text{threshold} \end{cases}$,
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  3. Discriminative model:

$$y(x) = P(C_k|x) = f(x),$$

with f an arbitrary function.

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- Also called *generalized linear models*
- Applicable if instead of x we use a vector of basis functions $\phi(x)$, corresponding to features in a feature space

Using regression for classification

- We can use a regression model, such as least squares to fit a linear classification model with a linear activation function

$$\min_{w,w_o} \sum_{i=1}^{\ell} (t_i - w^T x_i + w_0)^2,$$

where $t_i \in \{-1, 1\}$ is the label of the i-th training sample,

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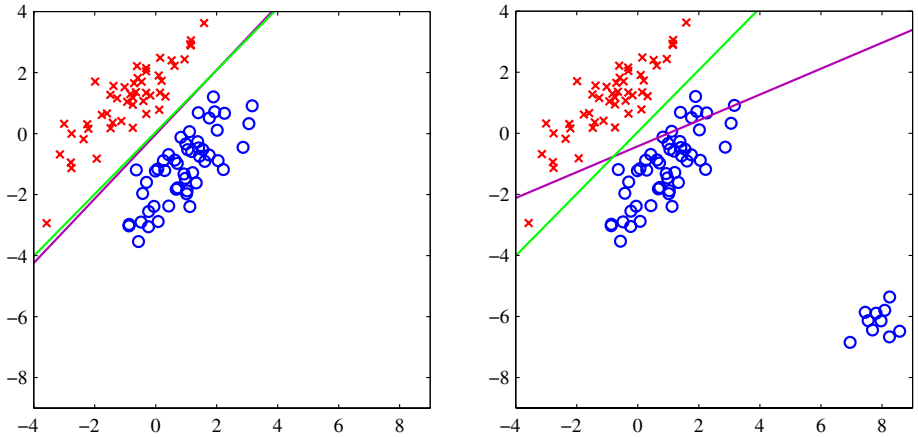
- but this strategy does not work well:



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- The precursor of neural networks
- Criticized by Marvin Minsky, producing a decline in research funding

Perceptron learning

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- Learning rule:

$$w^{(n)} = w^{(n-1)} + \eta(f_n - t_n)x_n$$

where $f_n = f(w^T x_n + w_0)$

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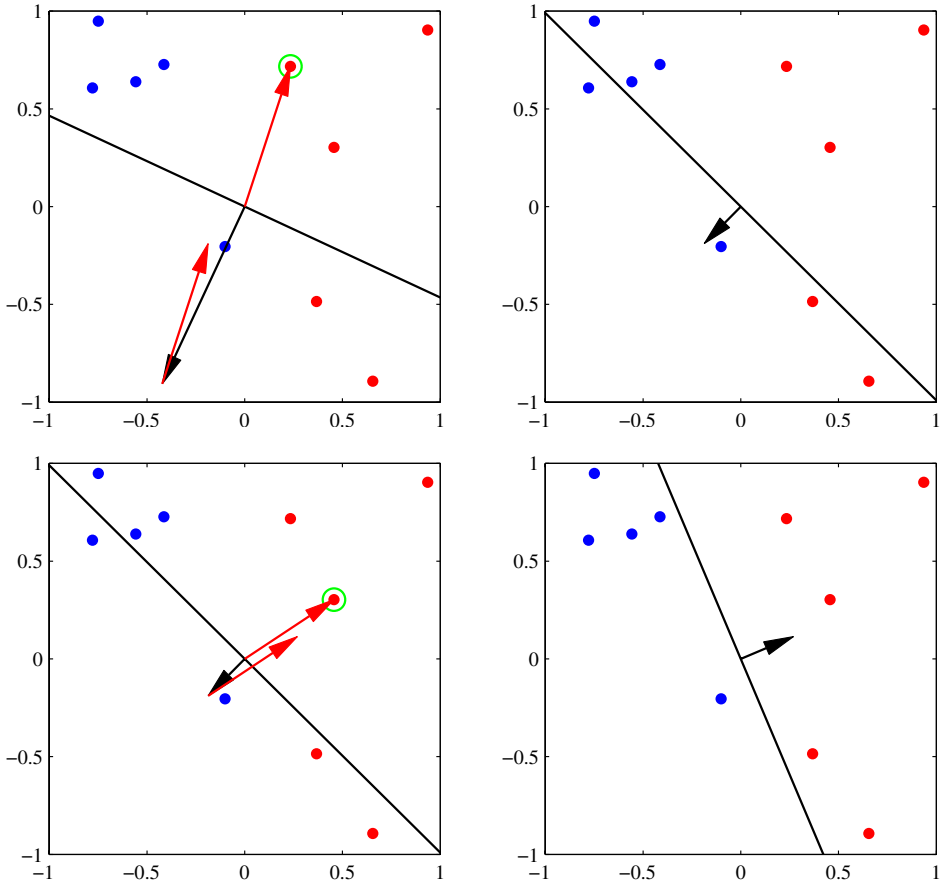

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- Non-convex optimization problem
- No convergence guarantee if samples are not linearly separable
- Its power may be increased by stacking several perceptron layers (multilayer perceptrons) and using smooth activation functions

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Parametric discrimination

- These three conditions are equivalent:
  - $P(C_1|x) \geq 0.5$
  - $\frac{P(C_1|x)}{1-P(C_1|x)} \geq 1$
  - $\text{logit}(P(C_1|x)) = \log \frac{P(C_1|x)}{1-P(C_1|x)} \geq 0$

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- If we assume that $P(x|C_1)$ and $P(x|C_2)$ are normally distributed sharing the same covariance matrix:

$$\text{logit}(P(C_1|x)) = \log \frac{P(C_1|x)}{P(C_2|x)} = w^T x + w_0,$$

where

$$w = \Sigma^{-1}(\mu_1 + \mu_2)$$

$$w_0 = -\frac{1}{2}(\mu_1 + \mu_2)^T \Sigma^{-1}(\mu_1 + \mu_2) + \log \frac{P(C_1)}{P(C_2)}$$

Logistic function

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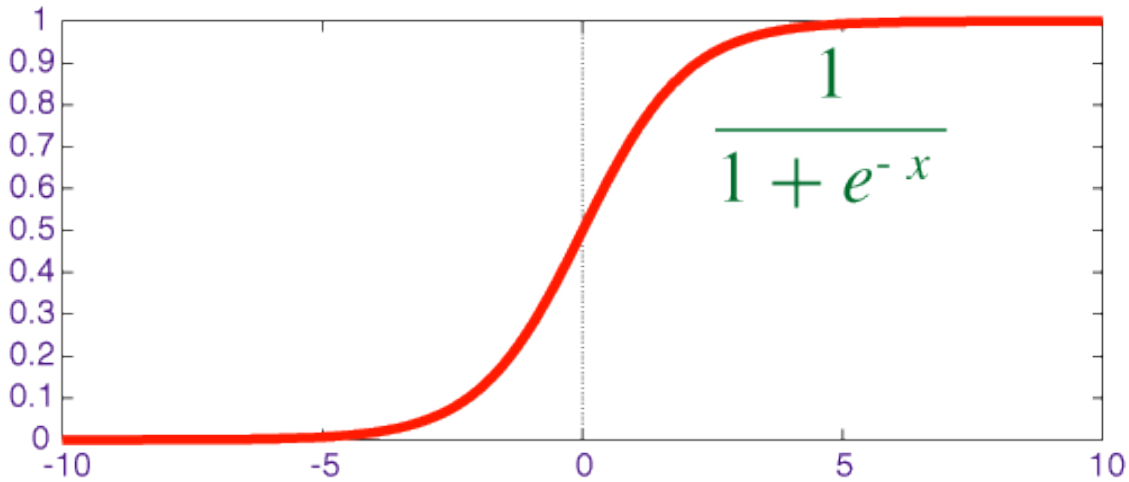
- σ is called the logistic or sigmoid function.



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- Cross-entropy error:

$$E(w) = -\ln p(\mathbf{t}|w) = -\sum_{n=1}^{\ell} [t_n \ln y_n + (1 - t_n) \ln(1 - y_n)]$$

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where $y_{nk} = y_k(x_n)$ and $\mathbf{T} \in \mathbb{R}^{\ell \times K}$ is a matrix of target variables with elements t_{nk}.

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- Multiclass cross-entropy error:

$$E(w_1, \dots, w_K) = -\ln p(\mathbf{T}|w_1 \dots w_K) = -\sum_{n=1}^{\ell} \sum_{k=1}^{K} t_{nk} \ln y_{nk}$$

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- $$w^{(\tau+1)} = w^{(\tau)} - \eta \sum_{n=1}^{\ell} (y_n - t_n)\phi_n$$

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  with $\mathbf{R}$ a diagonal matrix with $R_{nn} = y_n(1 - y_n)$.
- The resulting algorithm is called *iterative reweighted least squares.*

Regularization

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- Equivalent to the inclusion of a prior and finding a MAP solution for W.

Stochastic gradient descent

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- on-line gradient descent:

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 Alpaydin, E. 2010 Introduction to Machine Learning, 2Ed. The MIT Press. (Chap 10)

 Russell, S and Norvig, P. 2010 Artificial Intelligence: a Modern Approach, 3rd Ed, Prentice-Hall (Sect 18.6)