

LINEAR ALGEBRA Review

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# 1. Vector Space

- Defined over a field  $K$  ( $\mathbb{R}$  or  $\mathbb{C}$ )
- Vectors  $\in V$
- Scalars  $\in K$
- Operations

$$a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} a \cdot v_1 + w_1 \\ a \cdot v_2 + w_2 \\ a \cdot v_3 + w_3 \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

0 vector

$$\forall \psi \quad |\psi\rangle + 0 = |\psi\rangle$$

$| \rangle$  ket } Dirac Notation  
 $\langle |$  Bra } Bracket Notation.

## 2. Dirac Notation

Notation	Description
$z^*$	Complex conjugate of the complex number $z$ . $(1 + i)^* = 1 - i$ $i = \sqrt{-1}$
$ \psi\rangle$ <small>column</small>	Vector. Also known as a <i>ket</i> .
$\langle\psi $ <small>row</small>	Vector dual to $ \psi\rangle$ . Also known as a <i>bra</i> .
$\langle\varphi \psi\rangle$	Inner product between the vectors $ \varphi\rangle$ and $ \psi\rangle$ .
$ \varphi\rangle \otimes  \psi\rangle$	Tensor product of $ \varphi\rangle$ and $ \psi\rangle$ .
$ \varphi\psi\rangle$ <small><math> \varphi\rangle \psi\rangle</math></small>	Abbreviated notation for tensor product of $ \varphi\rangle$ and $ \psi\rangle$ .
$A^*$	Complex conjugate of the $A$ matrix.
$A^T$	Transpose of the $A$ matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the $A$ matrix, $A^\dagger = (A^T)^*$ . $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}.$
$\langle\varphi A \psi\rangle$	Inner product between $ \varphi\rangle$ and $A \psi\rangle$ . Equivalently, inner product between $A^\dagger \varphi\rangle$ and $ \psi\rangle$ .

### 3. Bases and linear independence

- Spanning Set

$$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \forall |v\rangle \in V \quad |v\rangle = \sum_i a_i |v_i\rangle$$

$$V = \mathbb{C}^2$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 |0\rangle + a_2 |1\rangle$$

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{a_1 + a_2}{\sqrt{2}} |v_1\rangle + \frac{a_1 - a_2}{\sqrt{2}} |v_2\rangle$$

- Linear dependent set

$$\{|v_1\rangle, \dots, |v_n\rangle\} \quad \exists a_1 \dots a_n \quad a_1 |v_1\rangle + \dots + a_n |v_n\rangle = 0$$

at least one  $a_i \neq 0$

- Basis: a non-linear dependent set that spans  $V$   
#elements of a basis = Dimension of the space

## 4. Linear operators and matrices

- Linear operator  $V, W$  vector spaces

$$A: V \rightarrow W \quad A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle)$$

$$A|v_i\rangle := A(|v_i\rangle)$$

Identity operator  $I_V |v\rangle = |v\rangle$

- Composition  $A: V \rightarrow W \quad B: W \rightarrow X$

$$B(A(|v\rangle)) =: (BA)(|v\rangle) = BA|v\rangle$$

- Matrix representation

$A: V \rightarrow W$   $\{|v_1\rangle, \dots, |v_n\rangle\}$  a basis for  $V$   
 $\{|w_1\rangle, \dots, |w_m\rangle\}$  a basis for  $W$

$$A(|v_i\rangle) = \sum_{j=1}^m A_{ij} |w_j\rangle \quad |v\rangle = a_1 |v_1\rangle + \dots + a_n |v_n\rangle$$

$$A(|v\rangle) = \sum_i A(a_i |v_i\rangle) = \sum_i \sum_j a_i A_{ij} |w_j\rangle$$

# S. Pauli Matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

## 6. Inner product

- Inner product  $(\cdot, \cdot): V \times V \rightarrow \mathbb{C}$

$$|v\rangle, |w\rangle \mapsto (|v\rangle, |w\rangle)$$

$$\langle v | w \rangle := (|v\rangle, |w\rangle)$$

1. Linear in the second argument

$$(|v\rangle, \sum_i \lambda_i |w_i\rangle) = \sum_i \lambda_i (|v\rangle, |w_i\rangle)$$

$$V = \mathbb{C}^n$$

$$\begin{aligned} & (v_1 \dots v_n, w_1, \dots, w_n) \\ &= \sum_i v_i^* w_i \end{aligned}$$

2.  $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$

3.  $(|v\rangle, |v\rangle) \geq 0$   $(|v\rangle, |v\rangle) = 0 \Leftrightarrow |v\rangle = 0$

- Dual vector

$$|v\rangle \Rightarrow \langle v| : V \rightarrow \mathbb{C}$$
$$|w\rangle \mapsto (|v\rangle, |w\rangle)$$

$$\langle v| (|w\rangle) = \langle v | w \rangle$$

- Inner product space  $\equiv$  Hilbert space if the dimension is finite

- orthogonality, norm

$|v\rangle$  and  $|w\rangle$  are orthogonal if  $\langle v | w \rangle = 0$

$\frac{|v\rangle}{\| |v\rangle \|}$  is normal

$|v\rangle$  is normal if  $\| |v\rangle \| = \langle v | v \rangle = 1$

- Orthonormal Basis

$\{|v_1\rangle, \dots, |v_n\rangle\}$  is a Basis, it is orthonormal if

$$\forall i \| |v_i\rangle \| = 1$$

$$\forall i \neq j \langle v_i | v_j \rangle = 0$$

$$\Leftrightarrow \langle v_i | v_j \rangle = \delta_{ij}$$

# - Gram-Schmidt

$\{|w_1\rangle, \dots, |w_d\rangle\}$  a basis

$$|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|}$$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^k \langle v_i | w_{k+1} \rangle |v_i\rangle}{\| |w_{k+1}\rangle - \sum_{i=1}^k \langle v_i | w_{k+1} \rangle |v_i\rangle \|}$$

$\{|v_1\rangle, \dots, |v_d\rangle\}$  is an orthonormal basis.

## - Outer product

$$|v\rangle \in V, |w\rangle \in W$$

$$|w\rangle\langle v| : V \rightarrow W$$

$$|v'\rangle \mapsto \langle v | v' \rangle |w\rangle$$

$$|w\rangle\langle v| (|v'\rangle) = |w\rangle\langle v | v' \rangle = \langle v | v' \rangle |w\rangle$$

$$\left[ |v\rangle\langle w| \right] \left[ |y\rangle\langle w| \right] = \langle w | y \rangle \langle w | x \rangle |v\rangle$$

$\{|1\rangle, |2\rangle, \dots, |d\rangle\}$  orthonormal basis

$$O = \sum_{i=1}^d |i\rangle\langle i| = I$$

$$O : V \rightarrow V$$

$$|v\rangle \mapsto O|v\rangle = \sum_{i=1}^d |i\rangle\langle i|v\rangle = \sum_{i=1}^d \overbrace{\langle i|v\rangle}^{v_i} |i\rangle = |v\rangle$$

$$|v\rangle = v_1|1\rangle + \dots + v_d|d\rangle$$

$$\langle i|v\rangle = v_1\langle i|1\rangle + \dots + v_i\langle i|i\rangle + \dots + v_d\langle i|d\rangle$$

$$= v_i$$



# 7. Eigenvectors and Eigenvalues (Sect. 2.1.5)

- Eigenvector

$$A|v\rangle = \lambda|v\rangle$$

Eigenvalue

Eigenvector

- Characteristic function

$$C(\lambda) \equiv \det |A - \lambda I| \quad C(\lambda) = 0$$

- Diagonal representation

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$A \equiv \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2|0\rangle\langle 0| + 2|1\rangle\langle 1|$$

## 8. Adjoint and Hermitian operators

- Adjoint

$$\langle v |, A |w\rangle = \langle A^\dagger |v\rangle, |w\rangle$$

↪ Adjoint of A

$$(AB)^\dagger = B^\dagger A^\dagger$$

A is matrix  $A^\dagger = (A^*)^T$

↪ Conjugate

- Hermitian or Self adjoint

$$A = A^\dagger$$

- Projector

W is a subspace of V

$|1\rangle, \dots, |d\rangle$  orthonormal basis for V,  $|1\rangle, \dots, |k\rangle$  o.b. for W.

$$P = \sum_{i=1}^k |i\rangle \langle i|$$

$$|v\rangle \in V \quad P|v\rangle \in W$$

- Normal

$$AA^\dagger = A^\dagger A$$

- Unitary

$$U^\dagger U = I \quad UU^\dagger = I$$

$$\begin{aligned} \langle u | v \rangle, \langle u | w \rangle &= \langle v | u^\dagger u | w \rangle \\ &= \langle v | I | w \rangle \\ &= \langle v | w \rangle \end{aligned}$$

## 9. Tensor Product

- Tensor product of spaces

- Tensor product properties

$$\begin{array}{l} V \\ W \\ V \otimes W \end{array}$$

$m$ -dimensional with basis  $\{|v_i\rangle\}_{i=1 \dots m}$   
 $n$ -dimensional with basis  $\{|w_j\rangle\}_{j=1 \dots n}$   
 $mn$ -dimensional.  
with basis  $\{|v_i\rangle \otimes |w_j\rangle\}_{\substack{i=1 \dots m \\ j=1 \dots n}}$

(1) For an arbitrary scalar  $z$  and elements  $|v\rangle$  of  $V$  and  $|w\rangle$  of  $W$ ,

$$z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle). \quad (2.42)$$

$$\begin{aligned} |v\rangle \otimes |w\rangle \\ \equiv |v, w\rangle \equiv |vw\rangle \end{aligned}$$

(2) For arbitrary  $|v_1\rangle$  and  $|v_2\rangle$  in  $V$  and  $|w\rangle$  in  $W$ ,

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle. \quad (2.43)$$

(3) For arbitrary  $|v\rangle$  in  $V$  and  $|w_1\rangle$  and  $|w_2\rangle$  in  $W$ ,

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle. \quad (2.44)$$

- Linear operators on Tensor product spaces

$A$  an operator in  $V$ ,  $B$  an operator on  $W$

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$$

- Kronecker Produkt

$$\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\ \vdots \\ v_m \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ \vdots \\ v_1 w_n \\ v_2 w_1 \\ \vdots \\ v_2 w_n \\ \vdots \\ v_m w_1 \\ \vdots \\ v_m w_n \end{bmatrix}$$

$$\underline{|01\rangle} = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|v\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

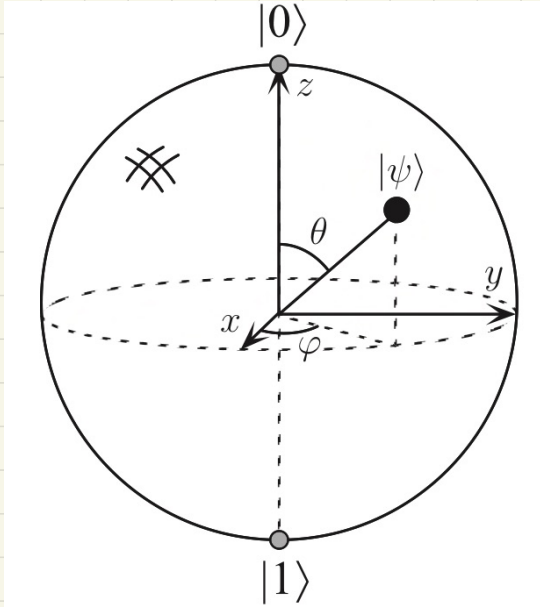
$$|v\rangle = v_1 |00\rangle + v_2 |01\rangle + v_3 |10\rangle + v_4 |11\rangle$$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

$$\alpha |01\rangle + \beta |10\rangle$$

$$|\alpha| + |\beta| = 1$$

# 10. Qubits and the Bloch Sphere (sect 1.2)



$$A: V \rightarrow W$$

$$\left\{ \begin{array}{l} \{v_1 \dots v_m\} \\ \{w_1 \dots w_n\} \end{array} \right\} \rightarrow \{ |v_1\rangle, |v_2\rangle \dots |v_m\rangle \}$$

$$A |v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

$\begin{array}{cc} \nearrow & \nearrow \\ w & v \end{array}$

$$\begin{aligned} A_{23} &= \langle w_2 | A (|v_3\rangle) \\ &= \langle w_2 | A |v_3\rangle \end{aligned}$$

$$\begin{aligned} & \langle w_2 | \overbrace{A |v_3\rangle} \\ &= \langle w_2 | \left( \sum_i A_{i3} |w_i\rangle \right) \\ &= \sum_i A_{i3} \langle w_2 | w_i \rangle \\ &= A_{23} \langle w_2 | w_2 \rangle \\ &= A_{23} \end{aligned}$$

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$$A = |v_j\rangle \langle v_k|$$

$$A_{lm} = \langle v_l | \overbrace{|v_j\rangle \langle v_k|}^A |v_m\rangle$$

$$A_{lm} = \begin{cases} 1 & \text{if } l=j \text{ and } m=k \\ 0 & \text{o.c.} \end{cases}$$