Qubits

QCP 2020-2

Fabio A. González

Universidad Nacional de Colombia

1. Quantum Mechanics Postulates

1. The superposition principle:

   what are the possible states of quantum system

2. The measurement principle:

   How much information from a quantum state we can access

3. Unitary Evolution:

   How a quantum system evolves through time

2. The superposition Principle

If a quantum system can be in one of two states it can also be in any linear combination of these states with complex coefficients.

$$|\alpha_i| = \sqrt{\alpha_i^* \alpha_i}$$

Possible states $\{|0\rangle, |1\rangle, \dots, |k-1\rangle\}$

Quantum state $\alpha_0 |0\rangle + \alpha_1 |1\rangle \dots \alpha_{k-1} |k-1\rangle$ $\alpha_i \in \mathbb{C}$ $\sum_{i=0}^{k-1} |\alpha_i|^2 = 1$

for $k=3$

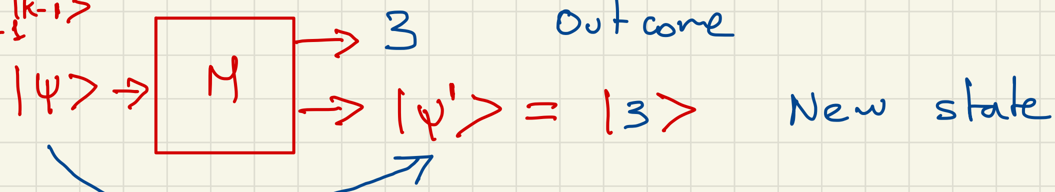
$$|\psi\rangle = |0\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle$$

3. The measurement Principle

- We can not measure the complex amplitudes α_i
- A measurement in a quantum system with k states produces k possible outcomes
- If we measure the system in the standard basis we get $|i\rangle$ with probability $|\alpha_i|^2$
- Measurement alters the state of the system, the new state is exactly the measurement outcome.

$|\psi\rangle = \alpha_0|0\rangle \dots \alpha_{k-1}|k-1\rangle$



Collapse of the wave function

- In a general measurement you select an orthonormal basis $\{|e_0\rangle, \dots, |e_{k-1}\rangle\}$.
- The outcome of the measurement is $|e_i\rangle$ with probability $|\beta_i|^2$ where β_i is the amplitude of $|e_i\rangle$ in the representation of ψ in the basis.

$$|\psi\rangle = \beta_0 |e_0\rangle + \dots + \beta_{k-1} |e_{k-1}\rangle$$

$$|\psi\rangle = \beta_0 |v\rangle + \beta_1 |v^\perp\rangle$$

$$\langle v|\psi\rangle = \beta_0$$

$$P_\psi(|v\rangle) = |\beta_0|^2 = |\langle v|\psi\rangle|^2$$

4. Qubits

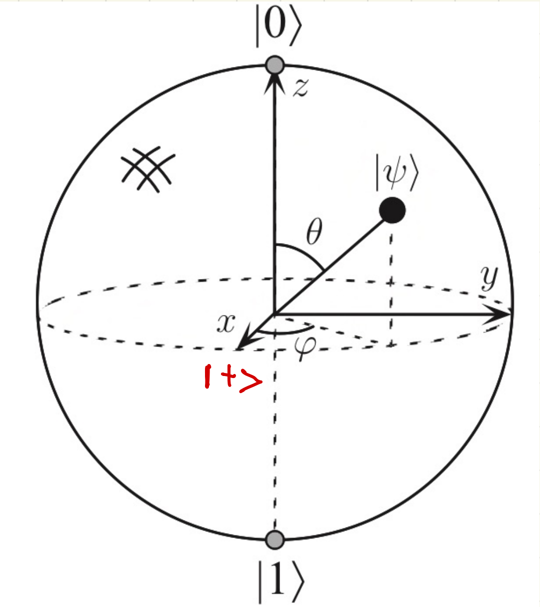
A qubit is a quantum system with two states.

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$|\alpha|^2$ probability of getting $|0\rangle$



A qubit can be represented in an arbitrary basis $\{|v\rangle, |w\rangle\}$ $\langle v|v\rangle = \langle w|w\rangle = 1$

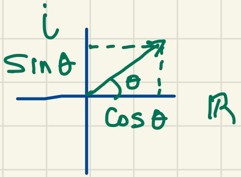
$\langle v|w\rangle = 0$

$$|\psi\rangle = \alpha'|v\rangle + \beta'|w\rangle$$

$\alpha = a e^{i\theta}$

magnitude direction

$e^{i\theta} = \cos\theta + i\sin\theta$



$$|\psi\rangle = a e^{i\gamma}|0\rangle + b e^{i\gamma'}|1\rangle$$

Global phase

$$= e^{i\gamma}\left(a|0\rangle + b e^{i(\gamma'-\gamma)}|1\rangle\right)$$

$$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\varphi}|1\rangle$$

$\theta = 2\arccos a$

Local phase

5. Phase Estimation

$$|\psi\rangle = \overbrace{\frac{1}{\sqrt{2}}}^{\alpha}|0\rangle + \overbrace{\frac{e^{i\theta}}{\sqrt{2}}}^{\beta}|1\rangle$$

$$|\psi\rangle \to \boxed{\begin{matrix}|0\rangle \\ |1\rangle\end{matrix}} \begin{matrix}\nearrow 0 \ \# n \\ \searrow 1 \ \# 100-n\end{matrix}$$

$$|\alpha|^2 = P(|0\rangle) \approx \frac{n}{100}$$

$$|\beta|^2 = P(|1\rangle) \approx \frac{100-n}{100}$$

Only gives information about magnitudes.

Is there any measurement that yields information about θ?

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$|\psi\rangle = \frac{1}{2}(|+\rangle + |-\rangle) + \frac{e^{i\theta}}{2}(|+\rangle - |-\rangle)$$

$$= \underbrace{\frac{1+e^{i\theta}}{2}}_{\alpha'}|+\rangle + \underbrace{\frac{1-e^{i\theta}}{2}}_{\beta'}|-\rangle$$

$$|\psi\rangle \to \boxed{\begin{matrix}|+\rangle \\ |-\rangle\end{matrix}} \begin{matrix}\nearrow + & |\alpha'| \\ \searrow - & |\beta'|\end{matrix}$$

6. General Qubit Bases

General basis $\{|v\rangle, |v^\perp\rangle\}$

$$|v\rangle = a|0\rangle + b|1\rangle$$
$$|v^\perp\rangle = b^*|0\rangle - a^*|1\rangle$$

$$|\psi\rangle^\dagger = \langle\psi|$$
$$\langle\psi|^\dagger = |\psi\rangle$$

$$\langle v^\perp|v\rangle = (b^*|0\rangle + a^*|1\rangle)^\dagger (a|0\rangle - b|1\rangle)$$
$$= (b\langle 0| + a\langle 1|)(a|0\rangle - b|1\rangle)$$
$$= ba\cancelto{1}{\langle 0|0\rangle} - b^2\cancelto{0}{\langle 0|1\rangle} + a^2\cancelto{0}{\langle 1|0\rangle} - ab\cancelto{1}{\langle 1|1\rangle} = 0$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$P_\psi(|v\rangle)$: Probability of measuring $|v\rangle$ when the state is ψ

$$P_\psi(|v\rangle) = |\langle v|\psi\rangle|^2$$
$$= |(a|0\rangle + b|1\rangle)^\dagger(\alpha|0\rangle + \beta|1\rangle)|^2$$
$$= |(a^*\langle 0| + b^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)|^2$$
$$= |a^*\alpha\langle 0|0\rangle + a^*\beta\cancel{\langle 0|1\rangle} + b^*\cancel{\langle 1|0\rangle} + b^*\beta\langle 1|1\rangle|^2$$
$$= |a^*\alpha + b^*\beta|^2$$

$$P_\psi(|v^\perp\rangle) = |b\alpha - \alpha\beta|^2$$

7. Unitary Operators

Quantum systems evolve through unitary operations (3rd postulate)

Unitary transformation $\longleftrightarrow$ rigid body motion $\longleftrightarrow$ doesn't change length $\longleftrightarrow$ preserve dot products

$$|0\rangle \longrightarrow \boxed{U} \longrightarrow |v_0\rangle = a|0\rangle + b|1\rangle$$
$$|1\rangle \longrightarrow \phantom{\boxed{U}} \longrightarrow |v_1\rangle = c|0\rangle + d|1\rangle$$

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad U^\dagger \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^*a + b^*b & a^*c + b^*d \\ c^*a + d^*b & c^*c + d^*d \end{pmatrix}$$

$$= \begin{pmatrix} \overset{1}{\cancel{\langle v_0|v_0\rangle}} & \overset{0}{\langle v_0|v_1\rangle} \\ \underset{0}{\cancel{\langle v_1|v_0\rangle}} & \underset{1}{\cancel{\langle v_1|v_1\rangle}} \end{pmatrix}$$

$$U^\dagger U = I = U U^\dagger$$

$$\text{Not} = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

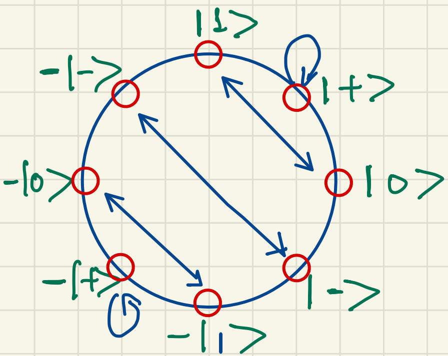
$$|+\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$X|0\rangle = |1\rangle \qquad X|+\rangle$$
$$X|1\rangle = |0\rangle \qquad X|-\rangle$$

$$X|+\rangle = |+\rangle \qquad X|-\rangle = -1|-\rangle$$

$$X|-\rangle = \frac{X}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\left(X|0\rangle - X|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right) = -1|-\rangle$$

$$X|-\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -1|-\rangle$$



Hadamard

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = |+\rangle \qquad H|1\rangle = |-\rangle$$

Phase flip

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$HZ = ?$$
$$ZH = ?$$

Rotation gate

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

y